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Unsteady flow through a circular tube

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The unsteady flow of viscous laminar incompressible fluid has been discussed for the velocity and temperature distributions and expressions for the Boussinesq coefficients k and k' have been calculated. It has been shown that for steady flow the well known result $k' = 2k$ can be calculated from the present expressions. The axial pressure gradient and the external rate of heat addition are taken as linear functions of time.

1. Velocity Distribution

The equation of motion is well known

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \qquad \text{.. (1)}$$

where the symbools have their usual meanings.

Assuming

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = f(t) \text{ the solution of (1) may be assumed}$$

$$u\,(r,t) = u_0(r)\,f(t) + u_1(r)\,f'(t) \qquad \text{...(2)}$$

where a dash denotes differentiation with respect to t.

The boundary conditions are

$$\left.\begin{aligned} &t > 0, \quad u = 0 \quad \text{On } r = a \\ &t > 0, \quad u = \text{finite} \quad \text{at } r = 0 \end{aligned}\right\} \qquad \text{...(3)}$$

where a is the radius of the tube.

Substituting (2) into (1), we have

$$f\left[1 + \nu\left(u_0'' + \frac{u_0'}{r}\right)\right] + f'\left[-u_0 + \nu\left(u_1'' + \frac{u_1'}{r}\right)\right] - u_1 f'' = 0 \qquad \text{...(4)}$$

Thus for all values of t, we get one of the relations as

$$\frac{df}{dt} = \text{constant} = A \text{ (say)} \qquad \text{...(5)}$$

$$\text{or } f = At + B$$

where A and B are certain constants.

Applying initial condition, we have

$$f\,(0) = B = \text{a constant.}$$

The expressions for u_0 and u_1 are obtained from (4) by equating the coefficients of $f(t)$ and $f'(t)$ to zero and solving it.

Thus

$$u_0 = \frac{1}{4\nu} (a^2 - r^2) \quad \text{.. (6)}$$

$$u_1 = \frac{a^2r^2}{16\nu^2} - \frac{1}{64\nu^2} (r^4 + 3a^4) \quad \text{...(7)}$$

$$u(r,t) = \frac{1}{4\nu} (a^2 - r^2)(At + B) + A\left[\frac{a^2r^2}{16\nu^2} + \frac{1}{64\nu^2} (r^4 + 3a^4)\right] \quad \text{...(8)}$$

DISCUSSION

The discharge of flux per second is

$$q = \int_0^a 2\pi\, u r dr = \frac{\pi}{8\nu}(At + B)a^4 - \frac{\Pi A a^6}{48\nu^2} \quad \text{...(9)}$$

The axial velocity is

$$U_1 = \frac{a^2}{4\nu} (At + B) - \frac{3Aa^4}{64\nu^2} \quad \text{.. (10)}$$

The maximum velocity, C is at $r = 0$, provided that

$$\frac{4\nu}{A} (At + B) > a^2 \quad \text{...(11)}$$

The expression for maximum velocity is same as U_1. Thus using the cross-section of tube, $A_1 = \pi a^2$, we may deduce the coefficients k and k' as introduced by Boussinesq.

These are connected by

$$q = UA_1 = kKA_1^2, \; C = k' K A_1 \quad \text{...(12)}$$

where U = average velocity, K = pressure gradient term $-\frac{1}{\mu}\frac{\partial p}{\partial z}$

and for the present problem, we have $K = f(t)/\nu$.

Thus we have in unsteady flow,

$$k = 0.03979 - \frac{a^2 A}{48\pi\nu (At + B)} \quad \text{..(13)}$$

$$k' = 0.07958 - \frac{3a^2 A}{64\pi\nu (At + B)} \quad \text{...(14)}$$

Hence, the well known result for steady flow

$$\frac{C}{U} = \frac{k'}{k} = 2 \qquad ...(15)$$

is obtained from (13) and (14) by putting $A = 0$.

2. Temperature Distribution

The energy equation (Pai 1956) is

$$\rho\, C_v \frac{\partial T}{\partial t} = \frac{\partial Q}{\partial t} + k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \phi \qquad ...(16)$$

Where the value of dissipation term, ϕ in this case is

$$\phi = \mu\left(\frac{\partial \mu}{\partial r}\right)^2 = \mu\left\{\frac{r^6}{\nu^4}\frac{A^2}{256} + r^4\left[\frac{A}{16\nu^3}(At+B) - \frac{A^2 a^2}{64\nu^4}\right]\right.$$
$$\left. + r^2\left[\frac{(At+B)^2}{4\nu^2} + \frac{A^2 a^4}{64\nu^4} - \frac{a^2 A(At+B)}{8\nu^3}\right]\right\} \qquad ...(17)$$

and $\frac{\partial Q}{\partial t}$ is the rate of external heat addition.

If we put

$$C_0 = \frac{A^2\mu'}{256\nu^4}, \quad \psi_1(t) = \frac{\mu' A}{16\nu^3}(At+B) - \frac{A^2\mu' a^2}{64\nu^4}$$

$$\psi_2(t) = \frac{\mu'(At+B)^2}{4\nu^2} + \frac{A^2\mu' a^4}{64\nu^4} - \frac{a^2\mu' A(At+B)}{8\nu^3}$$

the equation (16) without external rate of heat addition can be written

$$\frac{\partial T}{\partial t} = k'\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + [C_0 r^6 - \psi_1(t) r^4 + \psi_2(t) r^2] \qquad ...(18)$$

where

$$k' = \frac{k}{\rho c_v}, \quad \mu' = \frac{\mu}{\rho c_v}.$$

To solve equation (18), we assume

$$T(r,t) = r^6\phi_1(t) + r^4\phi_2(t) + r^2\phi_3(t) + \phi_4(t) \qquad ..(19)$$

Substituting (19) into (18) and equating the coefficients of various powers of r and constant term to zero, we have the following simple differential equations

$$\left.\begin{aligned} &\phi_1' - C_0 = 0 \\ &\phi_2' - 36k'\phi_1 - \psi_1 = 0 \\ &\phi_3' - 16k'\phi_2 - \psi_2 = 0 \\ &\phi_4' - 4k'\phi_3 = 0 \end{aligned}\right\} \qquad ..(20)$$

Initially $t = 0$ and $T = 0$ for all values of r.

Thus

$$\phi_1(0) = \phi_2(0) = \phi_3(0) = \phi_4(0) = 0 \qquad \text{...(21)}$$

Solving the equations of set (20) with boundary conditions of (21), we have

$$\phi_1 = -\frac{A^2\mu'}{256\nu^4}\,t \qquad \text{...(22a)}$$

$$\phi_2 = t^2\left[\frac{9A^2k'\mu'}{128\nu^4} + \frac{\mu'A^2}{32\nu^3}\right] + t\left[\frac{\mu'AB}{16\nu^3} - \frac{A^2\mu'a^2}{64\nu^4}\right] \qquad \text{.. (22b)}$$

$$\phi_3 = t^3\left[\frac{3A^2k'^2\mu'}{8\nu^4} + \frac{\mu'k'A^2}{6\nu^3} + \frac{\mu'A^2}{12\nu^2}\right]$$

$$+ t^2\left[\frac{\mu'k'AB}{2\nu^3} - \frac{A^2k'\mu'a^2}{8\nu^4} + \frac{\mu'AB}{4\nu^2} - \frac{a^2\mu'A^2}{16\nu^3}\right]$$

$$+ t\left[\frac{\mu'B^2}{4\nu^2} + \frac{A^2\mu'a^4}{64\nu^2} - \frac{a^2\mu'AB}{8\nu^3}\right] \qquad \text{...(22c)}$$

$$\phi_4 + t^4\left[\frac{3A^2k'^3\mu'}{8\nu^4} + \frac{\mu'k'^2A^2}{6\nu^3} + \frac{\mu'k'A^2}{12\nu^2}\right] \cdot$$

$$+ t^3\left[\frac{2\mu'k'^2AB}{3\nu^3} - \frac{A^2k'^2\mu a^2}{6\nu^4} + \frac{\mu'k'AB}{3\nu^2} - \frac{a^2\mu'k'A^2}{12\nu^3}\right]$$

$$+ t^2\left[\frac{\mu'k'B^2}{2\nu^2} + \frac{A^2\mu'k'a^4}{32\nu^3} - \frac{a^2\mu'k'AB}{4\nu^3}\right] \qquad \text{...(22d)}$$

Thus the temperature distribution comes out to be in the form of

$$T(r,t) = r^6(a_1t) + r^4(a_2t^2 + a_3t) + r^2(a_4t^3 + a_5t^2 + a_6t)$$
$$+ (a_7t^4 + a_8t^3 + a_9t^2) \qquad \text{.. (23)}$$

where $a_1 = 1, 2 \cdot 9$ are certain constants and are known from equations (22).

However, if the dissipation term is neglected as a small quantity, the energy equation is reduced to the form of equation of motion (Lahiri 1965) and can be easily solved. If the rate of external heat addition be taken in the form

$$\frac{1}{\rho c_v}\,\frac{\partial Q}{\partial t} = \phi_1(t), \text{ say} \qquad \text{...(24)}$$

the fourth equation of the set (20) with the form of $T(r,t)$ given by (19) is replaced by

$$\phi_4' - 4k'\,\phi_3 - \phi_1 = 0 \qquad \text{...(25)}$$

solution of which is

$$\phi_4(t) = a_7 t^4 + a_8 t^3 + \left(a_9 + \frac{a_1}{2}\right) t^2 \qquad \ldots(26)$$

It is easily seen that the axial temperature is given by $\phi_4(t)$.

Further, if in equation (21), we assume the initial boundary condition as $t = 0$, $T = f_1(r)$, we may write from (19)

$$\left.\begin{aligned} T(0,0) &= \phi_4(0) \\ T(a,0) &= a^6\phi_1(0) + a^4\phi_2(0) + (a^2\phi_3(0) + T(0,0) \\ T(a/2,0) &= \frac{a^6}{64}\phi_1(0) + \frac{a^4}{16}\phi_2(0) + \frac{a^2}{4}\phi_3(0) + T(0,0) \\ T(a/3,0) &= \frac{a^6}{729}\phi_1(0) + \frac{a^4}{81}\phi_2(0) + \frac{a^2}{9}\phi_3(0) + T(0,0) \end{aligned}\right\} \quad .(27)$$

where T (0,0) = initial axial temperature, T $(a, 0)$ = initial boundry temperature, T $(a/2, 0)$ = initial temperature at $r = a/2$, T $(a/3, 0)$ = initial temperature at $r = a/3$.

Solving the equation of set (27), we have for ϕ_1 (0) as

$$\phi_1(0) = \frac{1}{a^6}\left[36T\,(0,0) - \frac{3}{2}T(a,0) - \frac{69}{2}T\left(\frac{a}{3}, 0\right)\right] \ldots(28)$$

and similarly ϕ_2 (0), $\phi_3(0)$ may easily be calculated.

Using $\phi_1(0)$ in (21) and (22 a), we have

$$\phi_1(t) = \frac{A^2\mu' t}{256\nu^4} + \frac{1}{a^6}\left[36T\,(0,0) - \frac{3}{2}T(a,0) - \frac{69}{2}T\left(\frac{a}{3},0\right)\right] \ldots(29)$$

and thus from (20), we see that the calculations of ϕ_2, ϕ_3, ϕ_4 become more complicated. To avoid it, we have assumed the boundary conditions of (21).

If we assume in (19) that

$$T(r,t) = T_0(r)\,\psi_1(t) + T_1(r)\,\psi_2(r) + T_2(r)\psi_2'(t) \qquad \ldots(30)$$

then substituting in (18) and solving the equations obtained by equating to zero the coefficients of ψ_1, ψ_2, ψ_2'. we have

$$\left.\begin{aligned} T_0(r) &= T_0(a) + \frac{1}{36k'}\left(a^6 - r^6\right) \\ T_1(r) &= T_1(a) + \frac{1}{16k'}\left(a^4 - r_4\right) \\ T_2(r) &= T_2(a) - \frac{T_1(a)}{4k'}\left(a^2 - r^2\right) - \frac{a^4(a^2 - r^2)}{64k'^2} \\ &\quad + \frac{1}{576k'^2}\left(a^6 - r^6\right) \end{aligned}\right\} \quad \ldots(31)$$

where $T_0(a)$, $T_1(a)$, $T_2(a)$ are the xalues of $T_0(r)$, $T_1(r)$, $T_2(r)$ at $r = a$ *i. e.* boundary of the pipe.

From above calculations it is clear that $T\ (r, t)$ in both the cases is of the form $r^6 f_1(t) + r^4 f_2(t) + r^2 f_3(t) + f_4(t)$ where f_i *i.e.* are certain functions of time.

References

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